Transverse plasma waves and their instability

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Linearized equations are derived for disturbances in an infinite plasma without an imposed magnetic field. It is shown that besides the electrostatic, or longitudinal, waves which are usually considered, there can also exist electromagnetic, or transverse, waves. The two sets of waves are generally coupled, but one can nevertheless classify the waves as either mainly longitudinal or mainly transverse. It turns out that a plasma which is stable to longitudinal waves will be unstable to transverse waves unless the velocity distribution of its particles satisfies some rather stringent conditions. In a practical case these conditions would require the distribution to be isotropic.

1. Introduction

Plasma waves are in general not useful for the transmission of signals, but they are interesting because, for a wide variety of velocity distributions among the charged particles, they may exist with exponentially-growing amplitudes. Their occurrence then leads to a re-arrangement of particle velocities, presumably until a stable distribution is reached. Unstable waves thus tend to perform those functions which are fulfilled by molecular collisions in an ordinary gas.

In the past, attention has been paid mainly to electrostatic plasma waves. The criterion for their stability has been given by Penrose (1960), and their tendency to redistribute the particle velocities has been demonstrated by Buneman (1959). When an electrostatic wave is unstable its amplification rate is usually of the order of the plasma frequency, generally the fastest possible rate for any dynamical process in a plasma.

There is another form of plasma wave, in which the forces on the particles are electromagnetic, rather than electrostatic. This type of wave has not been so much discussed in the literature. However, the fact of its existence is implicit in some well-known equations for plasma waves, for example in equation (166) of chapter VI in *Plasma Physics* (Chandrasekhar 1960).

Weibel (1959) was the first to show that such waves can be unstable; the physical mechanism of the instability was then described by Fried (1959). Later Harris (undated MS.) discussed some further possibilities. All these authors treat only cases in which the undisturbed velocity distribution of the plasma particles has the form $C\exp\{-u^2/\sigma_\parallel^2-(v^2+w^2)/\sigma_\perp^2\}$. This restriction is not made in the present paper.

The waves in question are not be confused with the familiar fast electromagnetic waves, whose phase velocities are of the order of the speed of light, and which are simply a form of, say, radio waves, modified because the propagation occurs in a plasma rather than in a vacuum. The plasma waves to be discussed here have smaller phase velocities, which are usually complex and of the order of σ, the r.m.s. velocity dispersion of the charged particles which they affect. When such waves are unstable, the amplification rate is of the order σ/c times the plasma frequency, where c is the speed of light. Such instabilities are thus much less violent than electrostatic ones; if both types occur in a plasma, the latter are usually the more important. But much less stringent conditions are needed for the existence of an electromagnetic instability than for the existence of an electrostatic one. In fact, as Noerdlinger (1961) has shown, the conditions for a plasma to be electrostatically unstable are such as to prevent this instability from occurring in a number of rather interesting physical situations, for example in the case of a shock wave in the interplanetary medium. Here, and in many other cases, the transverse instabilities may become important.

2. The equations of motion

We shall treat only the case of an electron plasma without an imposed magnetic field. There is assumed to be present a smoothed-out background charge which takes no part in the motion, and merely serves to ensure overall space-charge neutrality. It is enough to allow for only one mobile species since, in a linear theory, a particle of charge Ze and mass m/μ is equivalent to μZ^2 particles of charge e and mass m.

In the undisturbed state let there be a uniform density of N particles per unit volume, each of charge e and mass m, and let $f_0(u,v,w)\,du\,dv\,dw$ be the proportion of particles whose velocities lie in the range $(u, u+du; v, v+dv; w, w+dw)$. Let the density of particles in this velocity range change to $N\{f_0 + f\exp[ik(x-at)]\}$ in the perturbed state. In general we take a to be a complex phase velocity, and k to be a positive real wave-number. The linearized form of the collision-free Boltzmann equation is then

$$ik(u-a)f + (e/m)(\mathbf{E} + c^{-1}\mathbf{u}\wedge\mathbf{H}).\nabla_u f_0 = 0, \tag{1}$$

where, for brevity,

$$\mathbf{u} \equiv (u,v,w) \quad \text{and} \quad \nabla_u \equiv (\partial/\partial u, \partial/\partial v, \partial/\partial w).$$

The electric and magnetic disturbance fields $\mathbf{E}$ and $\mathbf{H}$ can be expressed in terms of the potentials ϕ and $\mathbf{A}$ by the usual relations

$$\mathbf{E} = -\nabla\phi - c^{-1}\dot{\mathbf{A}}, \quad \mathbf{H} = \nabla\wedge\mathbf{A}. \tag{2}$$

The potentials are given in terms of the perturbed particle distribution by

$$\nabla^2\phi - c^{-2}\ddot{\phi} = -4\pi Ne\exp\{ik(x-at)\}\int f(\mathbf{u})\,d\mathbf{u},$$

or

$$k^2(1-a^2/c^2)\,\phi \equiv k^2\psi = 4\pi Ne\int f(\mathbf{u})\,d\mathbf{u}, \tag{3}$$

and

$$\nabla^2\mathbf{A} - c^{-2}\ddot{\mathbf{A}} = -(4\pi Ne/c)\exp\{ik(x-at)\}\int \mathbf{u}f(\mathbf{u})\,d\mathbf{u}$$

or

$$k^2(1-a^2/c^2)\,\mathbf{A} = (4\pi Ne/c)\int \mathbf{u}f(\mathbf{u})\,d\mathbf{u}. \tag{4}$$

The factor $\exp\{ik(x-at)\}$ has been omitted in (3) and (4), and will be dropped from now on. As usual

$$c^{-1}\dot{\phi} + \nabla.\mathbf{A} = 0$$

or

$$ac^{-1}\phi = A_x, \tag{5}$$

where $\mathbf{A} = (A_x, A_y, A_z)$. The electric and magnetic fields can therefore be conveniently expressed in terms of $\psi \equiv (1-a^2/c^2)\,\phi$, A_y and A_z. We have that

$$\left.\begin{aligned} E_x &= -ik(\phi - ac^{-1}A_x) = -ik(1-a^2/c^2)\,\phi \equiv -ik\psi,\\ E_y &= (ika/c)\,A_y, \quad E_z = (ika/c)\,A_z,\\ H_x &= 0, \quad H_y = -ikA_z, \quad H_z = ikA_y. \end{aligned}\right\} \tag{6}$$

In terms of the potentials, the linearized Boltzmann equation becomes

$$(u-a)f + \frac{e}{m}\left\{-\psi + \frac{1}{c}(vA_y + wA_z)\right\}\frac{\partial f_0}{\partial u} - \frac{e}{mc}(u-a)\left(A_y\frac{\partial f_0}{\partial v} + A_z\frac{\partial f_0}{\partial w}\right) = 0. \tag{7}$$

Three linear relations between ψ, A_y and A_z can be obtained from (3), (4) and (7). To find the first, divide (7) by $u-a$, and integrate over all velocities. Then, by (3),

$$\iiint f\,du\,dv\,dw = k^2\psi/4\pi Ne;$$

further, for any physically admissible velocity distribution,

$$\iiint_{-\infty}^{\infty}\frac{\partial f_0}{\partial v}du\,dv\,dw = \iint_{u,w}\{f_0(u,\infty,w) - f_0(u,-\infty,w)\}\,du\,dw$$
$$= 0,$$

and similarly for $\partial f_0/\partial w$. Finally, we define functions

$$I_{\alpha\beta\gamma\ldots}(a) = \iiint_{-\infty}^{\infty}\frac{\alpha\beta\gamma\ldots}{u-a}\frac{\partial f_0}{\partial u}du\,dv\,dw, \tag{8}$$

so that, for example, $$I_1(a) = \iiint_{-\infty}^{\infty}\frac{1}{u-a}\frac{\partial f_0}{\partial u}du\,dv\,dw,$$

$$I_{vv}(a) = \iiint_{-\infty}^{\infty}\frac{v^2}{u-a}\frac{\partial f_0}{\partial u}du\,dv\,dw,$$

and so on. Thus we find, from (7), that

$$\frac{k^2\psi}{4\pi Ne} - \frac{e}{m}I_1\psi + \frac{e}{m}\left(\frac{I_v}{c}A_y + \frac{I_w}{c}A_z\right) = 0$$

or $$(k^2 - \Omega^2 I_1)\,\psi + (\Omega^2/c)\,(I_v A_y + I_w A_z) = 0, \tag{9}$$

where $\Omega^2 = 4\pi Ne^2/m$.

Two further relations can be found on multiplying (7) by $v/(u-a)$ and $w/(u-a)$, respectively, and on integration over all velocities. In this case we have from (4) that

$$\left.\begin{aligned} &\iiint_{-\infty}^{\infty} vf(u,v,w)\,du\,dv\,dw = (ck^2/4\pi Ne)\,(1-a^2/c^2)\,A_y,\\ \text{and}\quad &\iiint_{-\infty}^{\infty} wf(u,v,w)\,du\,dv\,dw = (ck^2/4\pi Ne)\,(1-a^2/c^2)\,A_z. \end{aligned}\right\} \tag{10}$$

There also turn up integrals of the form

$$\iiint_{-\infty}^{\infty} v\frac{\partial f_0}{\partial v}du\,dv\,dw = \iint_{u,w}[vf_0]_{-\infty}^{\infty}\,du\,dw - \iiint_{-\infty}^{\infty} f_0\,du\,dv\,dw. \tag{11}$$

The first integral on the right-hand side vanishes here; the second equals unity, by the definition of f_0. Similarly,

$$\iiint_{-\infty}^{\infty} w\frac{\partial f_0}{\partial w}du\,dv\,dw = -1,$$

but
$$\iiint_{-\infty}^{\infty} w\frac{\partial f_0}{\partial v}du\,dv\,dw = \iiint_{-\infty}^{\infty} \frac{\partial(f_0 w)}{\partial v}du\,dv\,dw = 0,$$

and similarly for the integral of $v\,\partial f_0/\partial w$. Thus, from (7),

$$\frac{ck^2}{4\pi Ne}(1-a^2/c^2)A_y - \frac{e}{m}I_v\psi + \frac{e}{mc}(I_{vv}A_y + I_{vw}A_z) + \frac{e}{mc}A_y = 0,$$

or
$$-(\Omega^2/c)I_v\psi + [k^2(1-a^2/c^2) + k_0^2(I_{vv}+1)]A_y + k_0^2 I_{vw}A_z = 0, \tag{12}$$

and similarly

$$-(\Omega^2/c)I_w\psi + k_0^2 I_{vw}A_y + [k^2(1-a^2/c^2) + k_0^2(I_{ww}+1)]A_z = 0. \tag{13}$$

In (12) and (13) we have written $k_0 \equiv \Omega/c$. When ψ, A_y and A_z are eliminated from equations (9), (12) and (13), there remains the dispersion relation:

$$\begin{vmatrix} k^2-\Omega^2 I_1 & (\Omega^2/c)I_v & (\Omega^2/c)I_w \\ -(\Omega^2/c)I_v & k^2(1-a^2/c^2)+k_0^2(I_{vv}+1) & k_0^2 I_{vw} \\ -(\Omega^2/c)I_w & k_0^2 I_{vw} & k^2(1-a^2/c^2)+k_0^2(I_{ww}+1) \end{vmatrix} = 0. \tag{14}$$

An unstable plasma wave exists if and only if a real value of k may be found for which the phase velocity a has a positive imaginary part.

3. A discussion of the dispersion relation

The I-functions in (14) all have the form

$$I(a) = \int_{-\infty}^{\infty} \frac{1}{u-a}\frac{dg}{du}du, \tag{15}$$

and their properties are well known from the theory of electrostatic plasma waves. Thus $I(a)$ is clearly an analytic function of a in the upper (or in the lower) half-plane; if $g'(a_R)$ and $g''(a_R)$ exist it is readily shown that

$$\lim_{a\to a_R} I(a) = I(a_R) = J(a_R) + iK(a_R), \tag{16}$$

where
$$J(a_R) = \mathscr{P}\int_{-\infty}^{\infty} \frac{1}{u-a_R}\frac{dg}{du}du, \tag{17}$$

$$K(a_R) = \pi g'(a_R), \tag{18}$$

where a_R is real and where the limit is approached from the upper half-plane. The symbol $\mathscr{P}$ denotes 'principal part of'. (A proof is given in appendix 1.)

Suppose now that the zero velocity is chosen equal to the mean velocity of the particles, say. Let

$$\int_{-\infty}^{\infty} g(u)\,du = G;$$

then, if the modulus of the phase velocity $|a|$ much exceeds the half-width σ of the g-function,

$$I(a) = G \times O(1/a^2);$$

in particular, then, $I_{vv} = O(\sigma^2/a^2)$, $I_1 = O(1/a^2)$ and so on. On the other hand, if $|a|$ has the same order of magnitude as σ, or if it is much smaller, $I(a) = O(G/\sigma^2)$, and so $I_{vv} = O(1)$, $I_1 = O(1/\sigma^2)$, and so on.

For waves with a phase velocity of order c, we have $\sigma \ll |a|$ in the plasmas ordinarily encountered. When terms of the order of σ^2/c^2 are neglected, the determinant in (14) reduces to its diagonal terms only; the dispersion relation then becomes either

$$k^2 = \Omega^2/a^2,$$

which describes an electrostatic wave of very small wave-number, or

$$k^2a^2/c^2 \equiv \omega^2/c^2 = k^2 + k_0^2, \tag{19}$$

which is the familiar dispersion relation for electromagnetic radiation in a plasma.

But here we are interested, rather, in waves for which the modulus of the phase velocity is small compared with the speed of light, and for which we can replace by unity the factors $(1 - a^2/c^2)$, occurring in the diagonal terms of (14); the dispersion relation now becomes:

$$\begin{vmatrix} k^2 - \Omega^2 I_1 & (\Omega^2/c)\, I_v & (\Omega^2/c)\, I_w \\ -(\Omega^2/c)\, I_v & k^2 + k_0^2(1 + I_{vv}) & k_0^2\, I_{vw} \\ -(\Omega^2/c)\, I_w & k_0^2\, I_{vw} & k^2 + k_0^2(1 + I_{ww}) \end{vmatrix} = 0. \tag{20}$$

If I_v and I_w vanish, as may happen in cases with sufficient symmetry, then the electrostatic and the electromagnetic parts of the disturbance uncouple; the electrostatic part of the dispersion relation takes its usual form

$$k^2 - \Omega^2 I_1(a) = 0. \tag{21}$$

The order of magnitude of k for waves of interest is then given by

$$k^2 = O(\Omega^2/a^2). \tag{22}$$

The electromagnetic part of the dispersion relation becomes

$$\begin{vmatrix} k^2 + k_0^2(1 + I_{vv}) & k_0^2 I_{vw} \\ k_0^2\, I_{vw} & k^2 + k_0^2(1 + I_{ww}) \end{vmatrix} = 0,$$

that is $$(k^2 + k_0^2)^2 + 2(k^2 + k_0^2)\, k_0^2(I_{vv} + I_{ww}) + k_0^4(I_{vv}\, I_{ww} - I_{vw}^2) = 0,$$

or $$\lambda^2 + 2\lambda(I_{vv} + I_{ww}) + (I_{vv}\, I_{ww} - I_{vw}^2) = 0, \tag{23}$$

where $\lambda = 1 + k^2/k_0^2$. For interesting values of the phase velocity, I_{vv}, I_{vw} and I_{ww} are all of order unity; the corresponding values of k are then of order $k_0 \equiv \Omega/c$, and so are much smaller than the typical wave-number for an electrostatic wave with a comparable phase velocity.

The terms I_v and I_w do not, in general, vanish, and introduce a coupling between the electrostatic and electromagnetic modes. This affects the electrostatic modes only slightly. Here we expect k^2 to be of order Ω^2/σ^2, much larger than $k_0^2 \equiv \Omega^2/c^2$. The dispersion relation (20) therefore reduces, approximately, to

$$\begin{vmatrix} k^2-\Omega^2 I_1 & (\Omega^2/c)\,I_v & (\Omega^2/c)\,I_w \\ -(\Omega^2/c)\,I_v & k^2 & 0 \\ -(\Omega^2/c)\,I_w & 0 & k^2 \end{vmatrix} = 0; \tag{24}$$

when k^2 is set equal to $\Omega^2 I_1$ in the second and third diagonal terms, as may be done to the accuracy required, the relation (24) becomes

$$(k^2-\Omega^2 I_1)\,\Omega^4 I_1^2 + (\Omega^4/c^2)\,\Omega^2 I_1 (I_v^2+I_w^2) = 0$$

or

$$k^2-\Omega^2 I_1 = -(\Omega^2/c^2)\,(I_v^2+I_w^2). \tag{25}$$

An additional term, of the order of σ^2/c^2 times the dominant terms, has now been introduced. This correction is negligible. The usual form for the electrostatic dispersion relation is thus completely adequate. But the coupling terms make an important change in the dispersion relation for the electromagnetic mode. Here the value of k^2 for the interesting range of phase velocities will be of the order of Ω^2/c^2, which is small in comparison with $\Omega^2 I_1 = O(\Omega^2/\sigma^2)$. Thus the dispersion relation is now, to a good enough approximation,

$$\begin{vmatrix} -\Omega^2 I_1 & (\Omega^2/c)\,I_v & (\Omega^2/c)\,I_w \\ -(\Omega^2/c)\,I_v & k^2+k_0^2(1+I_{vv}) & k_0^2 I_{vw} \\ -(\Omega^2/c)\,I_w & k_0^2 I_{vw} & k^2+k_0^2(1+I_{ww}) \end{vmatrix} = 0, \tag{26}$$

or, after some re-arrangement and with $\lambda = 1+k^2/k_0^2$,

$$\lambda^2+\lambda(\mathscr{I}_{vv}+\mathscr{I}_{ww})+\mathscr{I}_{vv}\mathscr{I}_{ww}-\mathscr{I}_{vw}^2 = 0. \tag{27}$$

In this relation we have written

$$\mathscr{I}_{vv} = I_{vv}-I_v^2/I_1, \quad \mathscr{I}_{vw} = I_{vw}-I_v I_w/I_1, \quad \mathscr{I}_{ww} = I_{ww}-I_w^2/I_1. \tag{28}$$

It is satisfactory that the dispersion relations (21) and (27) are both invariant, in a Galilean sense, with respect to changes in velocity of the frame of reference. Invariance with respect to changes in the u-component is assured, since this enters into the expressions for the I-functions only via the term $u-a$, which is a difference between two velocity components. To see that $\mathscr{I}_{vv}$ is invariant with respect to a change in the v-component of velocity of the frame of reference, substitute $v' = v+V$; then, using the obvious notation,

$$I'_{vv} = I_{vv}+2VI_v+V^2 I_1, \quad I'_v = I_v+VI_1 \quad \text{and} \quad I'_1 = I_1,$$

but

$$I'_{vv}-I'^2_v/I'_1 = I_{vv}-I_v^2/I_1 \equiv \mathscr{I}_{vv}.$$

Similarly, $\mathscr{I}_{vw}$ and $\mathscr{I}_{ww}$ may be shown to be invariant. It is also readily seen that $\mathscr{I}_{vv}+\mathscr{I}_{ww}$ and $\mathscr{I}_{vv}\mathscr{I}_{ww}-\mathscr{I}_{vw}^2$, both of them expressions which occur in (27), are invariant with respect to a rotation of the v- and w-axes about the u-axis.

4. Sufficient conditions for the instability of a plasma

We now establish that electromagnetic instabilities do occur among a rather wide class of velocity distributions. For the sake of simplicity we shall here discuss only distributions with central symmetry, that is those for which a standard of rest can be found such that the undisturbed velocity distribution function satisfies

$$f_0(u,v,w) = f_0(-u,\ -v,\ -w). \tag{29}$$

The corresponding values of $I_1(a)$, $I_{vv}(a)$, $I_{vw}(a)$ and $I_{ww}(a)$ are then real and $I_v(a)$ and $I_w(a)$ are pure imaginary when a is pure imaginary. To establish this, note that

$$I_{vv}(i\eta) = \iiint_{-\infty}^{\infty} \frac{v^2}{u-i\eta}\frac{\partial f_0}{\partial u}\,du\,dv\,dw = \iiint_{-\infty}^{\infty} \frac{v^2}{u+i\eta}\frac{\partial f_0}{\partial u}\,du\,dv\,dw, \tag{30}$$

in virtue of (29); thus $I_{vv}(i\eta) = I^*_{vv}(i\eta)$, and the required result follows. The proof for the other I-functions is analogous.

Now equation (27), which is a form of the dispersion relation for transverse waves, may be written

$$[\lambda + \tfrac{1}{2}(\mathscr{I}_{vv}+\mathscr{I}_{ww})]^2 = \tfrac{1}{4}(\mathscr{I}_{vv}-\mathscr{I}_{ww})^2 + \mathscr{I}^2_{vw}. \tag{31}$$

For a centrally symmetrical distribution all coefficients in this equation are real when a is pure imaginary, and, in particular, the right-hand side is necessarily non-negative.

A sufficient condition for a transverse instability to occur is that a real positive wave-number k shall correspond to some phase velocity with a positive imaginary part. Since, by definition, $\lambda = 1 + k^2/k_0^2$ this means that there must exist a phase velocity in the upper half-plane whose corresponding λ is real and larger than unity. By equation (31) this is so if $\mathscr{I}_{vv}+\mathscr{I}_{ww}$ is real and less than -2 somewhere on the imaginary axis in the upper half-plane, and therefore, by continuity, if

$$\mathscr{I}_{vv}+\mathscr{I}_{ww} < -2$$

for $a = 0$. Now

$$\mathscr{I}_{vv}+\mathscr{I}_{ww} = I_{vv}+I_{ww} - (1/I_1)\,(I_v^2+I_w^2). \tag{32}$$

All cases can be excluded from this discussion in which I_1 is real and positive anywhere in the upper half-plane, since it follows from (21) that the corresponding plasma will be electrostatically unstable, and that an electrostatic instability will always be faster than an electromagnetic one. Therefore, in the cases of interest here, both I_1 and $I_v^2+I_w^2$ will be negative on $a = i\eta$, the imaginary axis; it follows from (32) that on this axis

$$\mathscr{I}_{vv}+\mathscr{I}_{ww} \leqslant I_{vv}+I_{ww}, \tag{33}$$

with equality only if $I_v = I_w = 0$.

We shall now work out an expression for

$$I_{tt}(0) = I_{vv}(0)+I_{ww}(0) = \iiint_{-\infty}^{\infty} \frac{v^2+w^2}{u}\frac{\partial f_0}{\partial u}\,du\,dv\,dw. \tag{34}$$

The central symmetry of f_0 ensures that there is no divergence near $u = 0$. To save confusion in the notation the suffix zero of the undisturbed velocity distribution function is dropped from here on. In terms of spherical polar co-ordinates (U,θ,φ) with the initial line $\theta = 0$ along the positive u-axis, we find that

$$I_{tt}(0) = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} U\frac{\sin^2\theta}{\cos\theta}\left(\cos\theta\frac{\partial f}{\partial U} - \frac{\sin\theta}{U}\frac{\partial f}{\partial \theta}\right) U^2\sin\theta\,dU\,d\theta\,d\varphi. \tag{35}$$

Put
$$\mu = \cos\theta$$
and
$$\int_0^{2\pi} f(U,\mu,\varphi)\,d\varphi = 2\pi h(U,\mu); \tag{36}$$
then
$$I_{tt}(0) = 2\pi\int_0^\infty\int_{-1}^1\left[(1-\mu^2)\,U\frac{\partial h}{\partial U}+\frac{(1-\mu^2)^2}{\mu}\frac{\partial h}{\partial\mu}\right]U^2\,dU\,d\mu. \tag{37}$$
Now expand h in terms of Legendre polynomials
$$h(U,\mu) = \sum_{n=0}^\infty h_{2n}(U)\,P_{2n}(\mu); \tag{38}$$
the condition of central symmetry ensures that only polynomials of even order occur in this expansion. On defining
$$H_{2n} = \int_0^\infty h_{2n}(U)\,U^2\,dU, \tag{39}$$
one can show (see appendix 2) that
$$I_{tt}(0) = -8\pi H_0 + \sum_{n=1}^\infty i_n H_{2n}, \tag{40}$$
where
$$i_n = (-1)^{n-1}2^{2n+1}(n!)^2/(2n)!.$$
Now the normalization condition on the undisturbed distribution function is such that
$$1 = \int_0^\infty\int_{-1}^1\int_0^{2\pi} f(U,\mu,\varphi)\,U^2\,dU\,d\mu\,d\varphi$$
$$= 2\pi\int_{-1}^1 H_0\,d\mu = 4\pi H_0;$$
hence
$$I_{tt}(0) = -2 + \sum_{n=1}^\infty i_n H_{2n}. \tag{41}$$
To avoid a transverse instability certainly requires that $I_{tt}(0) \geqslant -2$. This will be the case if the sum $\sum_{n=1}^\infty i_n H_{2n}$ on the right-hand side of (41) never becomes negative.

The calculation so far has been confined to one particular direction for the u-axis, that is to one direction of the wave-normal $\mathbf{n}$. But if the plasma is to be stable, $I_{tt} \geqslant -2$ whatever the direction chosen for $\mathbf{n}$. Therefore we define a fundamental initial line with respect to which a particular wave-vector points into the direction (α_0, β_0). Let the velocity distribution function be written
$$f(\mathbf{u}) \equiv f(U,\alpha,\beta) = \psi_0(U) + \sum_{n=1}^\infty\sum_{m=0}^{2n}\psi_{2n}^{(m)}(U)\,S_{2n}^{(m)}(\alpha,\beta), \tag{42}$$
when expressed with respect to this fundamental initial line, where the $S_{2n}^{(m)}$ are spherical harmonics. Again only harmonics of even order occur, owing to the requirement for central symmetry. Expressed with respect to an initial line which is parallel to a wave-normal $\mathbf{n}$ pointing into the direction (α_0, β_0), the distribution function may be expressed as
$$f(\mathbf{u}) = \sum_{n=0}^\infty\sum_{m=0}^{2n} f_{2n}^{(m)}(U)\,p_{2n}^m(\mu)\cos(m\varphi+\epsilon_{m,n}), \tag{43}$$

where the $\epsilon_{m,n}$ are suitable constants. On equating harmonics of equal order in (42) and (43) one finds that

$$\sum_{m=0}^{2n} \psi_{2n}^{(m)}(U)\, S_{2n}^{(m)}(\alpha,\beta) = \sum_{m=0}^{2n} f_{2n}^{(m)}(U)\, p_{2n}^{m}(\mu)\cos(m\varphi+\epsilon_{m,n}). \tag{44}$$

But $\mu = 1$ when $\alpha = \alpha_0$ and $\beta = \beta_0$; further $p_{2n}^{0}(1) \equiv P_{2n}(1) = 1$, while $p_{2n}^{m}(1) = 0$ for all non-zero m. Therefore

$$\sum_{m=0}^{2n} \psi_{2n}^{(m)}(U)\, S_{2n}^{(m)}(\alpha_0,\beta_0) = f_{2n}^{(0)}(U). \tag{45}$$

It follows from (36) and (38) that

$$h_{2n}(U) \equiv f_{2n}^{(0)}(U) = \sum_{m=0}^{2n} \psi_{2n}^{(m)}(U)\, S_{2n}^{(m)}(\alpha_0,\beta_0). \tag{46}$$

On defining
$$\Psi_{2n}^{(m)} = \int_0^\infty \psi_{2n}^{(m)}(U)\, U^2\, dU,$$

one deduces that
$$H_{2n} = \sum_{m=0}^{2n} \Psi_{2n}^{(m)}\, S_{2n}^{(m)}(\alpha_0,\beta_0), \tag{47}$$

and from (41) that
$$I_{tt}(0) = -2 + \sum_{n=1}^{\infty} \sum_{m=0}^{2n} i_n \Psi_{2n}^{(m)} S_{2n}^{(m)}(\alpha_0,\beta_0). \tag{48}$$

Now the average value of any spherical harmonic, of order one or larger, over a sphere is zero. Therefore, if the double sum in (48) must not become negative for any value of (α_0,β_0), then it must vanish for all (α_0,β_0). Further, none of the i_n coefficients in (48) vanishes. Therefore it follows that all the $\Psi_{2n}^{(m)}$'s vanish, for $n \geqslant 1$. We return to (42) and see that this implies that

$$\int_0^\infty U^2 f(U,\alpha,\beta)\, dU = \Psi_0, \tag{49}$$

independent of α and β. This integral is proportional to the number of particles in the undisturbed plasma moving into a unit solid angle around the direction (α,β). Thus, *in order that $I_{tt}(0) \geqslant -2$ for all directions of the wave-vector we require that the number of particles in the plasma moving into any given solid angle shall be independent of direction.* In fact, when this is the case, $I_{tt}(0) = -2$ for all directions of the wave-normal, and it can easily be shown that $I_{vv} = I_{ww} = -1$ and $I_{vw} = 0$.

Thus the velocity distribution in the plasma must satisfy some rather stringent conditions if instability is to be avoided. In fact the requirement can be made still stronger, for (33) states that

$$\mathscr{I}_{vv} + \mathscr{I}_{ww} \leqslant I_{vv} + I_{ww},$$

with equality only if $I_v = I_w = 0$. Now, at best, the right-hand side equals -2 for all directions of the wave-normal. Therefore I_v and I_w must vanish for all wave-normals in order that $\mathscr{I}_{vv} + \mathscr{I}_{ww}$ shall never be smaller than -2 and that no instability shall occur. Both I_v and I_w are pure imaginary at $u = 0$, so that

$$I_v = \pi i \iint v\left(\frac{\partial f}{\partial u}\right)_{u=0} dv\, dw \tag{50}$$

$$= -\pi i \int_0^\infty \int_0^{2\pi} \left(\frac{\partial f}{\partial \theta}\right)_{\theta=\frac{1}{2}\pi} U\cos\varphi\, dU\, d\varphi, \tag{51}$$

expressed in polar co-ordinates. Similarly,

$$I_w = -\pi i \int_0^\infty \int_0^{2\pi} \left(\frac{\partial f}{\partial \theta}\right)_{\theta=\frac{1}{2}\pi} U \sin\varphi \, dU \, d\varphi. \quad (52)$$

Both expressions must vanish for all directions of the wave-normal. It is shown in appendix 2 that this means that the distribution function must also be such that

$$\int_0^\infty U f(U, \alpha, \beta) \, dU = \Xi_0, \quad (53)$$

independent of α and β.

The argument shows that no disturbance with a real, non-zero wave-number and zero phase velocity will exist in a plasma whose velocity distribution function satisfies conditions (49) and (53). We go on to prove that in such cases no disturbance with a real wave-number can exist with an imaginary phase velocity $a = i\eta$ where η is small. To establish this consider the values of $\mathscr{I}_{vv}$, $\mathscr{I}_{ww}$ and $\mathscr{I}_{vw}$ at $a = i\eta$. We have that

$$\mathscr{I}_{vv}(i\eta) = \mathscr{I}_{vv}(0) + i\eta(\partial \mathscr{I}_{vv}/\partial\eta)_0, \quad (54)$$

approximately, that

$$\left(\frac{\partial \mathscr{I}_{vv}}{\partial \eta}\right)_0 = \left(\frac{\partial I_{vv}}{\partial \eta}\right)_0 - \left(\frac{\partial}{\partial \eta}\frac{I_v^2}{I_1}\right)_0 = \left(\frac{\partial I_{vv}}{\partial \eta}\right)_0, \quad (55)$$

since $I_v(0) = 0$; and if $I_{vv} = J_{vv} + iK_{vv}$, expressed in real and imaginary parts, then

$$\left(\frac{\partial I_{vv}}{\partial \eta}\right)_0 = \left(\frac{\partial J_{vv}}{\partial \eta}\right)_0 = -\left(\frac{\partial K_{vv}}{\partial u}\right)_0. \quad (56)$$

But

$$K_{vv}(u) = \pi \int_{-\infty}^\infty \int_{-\infty}^\infty v^2 \frac{\partial f}{\partial u} \, dv \, dw,$$

and so

$$\left(\frac{\partial K_{vv}}{\partial u}\right)_0 = \pi \int_{-\infty}^\infty \int_{-\infty}^\infty v^2 \left(\frac{\partial^2 f}{\partial u^2}\right)_0 dv \, dw. \quad (57)$$

In terms of spherical polars, with $\theta = 0$ parallel to the positive u-axis,

$$\left(\frac{\partial^2}{\partial u^2}\right)_{u=0} = \frac{1}{U^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{U}\frac{\partial}{\partial U};$$

thus

$$\left(\frac{\partial K_{vv}}{\partial u}\right)_0 = \pi \int_0^\infty \int_0^{2\pi} \left(\frac{1}{U^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{U}\frac{\partial f}{\partial U}\right) U^3 \cos^2\varphi \, dU \, d\varphi$$

$$= \pi \int_0^\infty \int_0^{2\pi} \left(\frac{\partial^2}{\partial \theta^2} - 2\right) f U \cos^2\varphi \, dU \, d\varphi. \quad (58)$$

It now easily follows that, if a distribution function satisfies condition (53), then only its isotropic part makes any contribution to the integral in (58). Hence, in such cases,

$$\left(\frac{\partial K_{vv}}{\partial u}\right)_0 = -2\pi^2 \int_0^\infty f U \, dU < 0, \quad \text{and} \quad \left(\frac{\partial J_{vv}}{\partial \eta}\right)_0 = -\left(\frac{\partial K_{vv}}{\partial u}\right)_0 > 0.$$

Thus both $\mathscr{I}_{vv}(i\eta)$ and $\mathscr{I}_{ww}(i\eta)$ exceed -1. Further, it is readily proved that $\mathscr{I}_{vw}(i\eta) = 0$ now. The equation for λ becomes simply

$$[\lambda + \tfrac{1}{2}(\mathscr{I}_{vv} + \mathscr{I}_{ww})]^2 = 0. \quad (59)$$

Since $\mathscr{I}_{vv}(i\eta)+\mathscr{I}_{ww}(i\eta) > -2$, the value of λ corresponding to $a = i\eta$ is less than unity, and the corresponding wave number is not real. Thus no unstable solutions exist near the zero phase velocity in this case.

5. Conclusion

It has been shown that unstable transverse waves can occur in plasmas with a wide variety of velocity distribution functions. When these functions have central symmetry an instability will certainly arise unless conditions (49) and (53) hold, that is unless the number of particles moving into a solid angle of given size and their harmonic mean velocity are independent of direction. If these conditions hold this does not mean, strictly speaking, that the velocity distribution has to be isotropic; to ensure isotropy would require that all integrals

$$F_n = \int_0^\infty U^n f\, dU \quad (n = 0, 1, 2, 3, \ldots)$$

shall have values independent of orientation. In particular, it is not required by conditions (49) and (53) that there shall be pressure isotropy; a condition on the integral F_4 would be needed to make a statement about the pressure. But in practice a plasma would need to be specially prepared if its velocity distribution were to satisfy (49) and (53) and it were yet anisotropic; this can hardly be expected of the distributions likely to be met with in a practical problem.

Two important questions are not resolved by our discussion. We cannot say that a plasma will definitely be stable if its distribution function satisfies (49) and (53) since we have not considered what values of k^2 correspond to phase velocities considerably different from zero. Further, it has not been possible to obtain any results for velocity distributions without central symmetry. The author believes, but has been unable to prove, that all such distributions are unstable.

In a paper to follow some typical unstable velocity distributions are considered which might arise in plausible physical situations, and the development of the instability is sketched.

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Appendix 1: General properties of the *I*-functions

We define
$$I(a) = \int_{-\infty}^{\infty} \frac{g'(u)}{u-a}\,du \tag{A. 1.1}$$
and show first that $I(a)$ is analytic in the upper half-plane. Since the zero velocity can be fixed arbitrarily, we consider only a point $a = i\eta$ on the imaginary axis. Now
$$I(i\eta) = \int_{-\infty}^{\infty} \frac{g'(u)}{u-i\eta}\,du = \int_{-\infty}^{\infty} \frac{g(u)}{(u-i\eta)^2}\,du, \tag{A. 1.2}$$
after integration by parts, since $g(\pm\infty) = 0$. Further, for all physically admissible functions
$$\int_{-\infty}^{\infty} |g(u)|\,du = M, \quad \text{say}, \quad < \infty,$$
and so
$$|I(i\eta)| \leqslant M/\eta^2. \tag{A. 1.3}$$
Thus I can have no singularities in the upper half-plane. Further,
$$I'(i\eta) = 2\int_{-\infty}^{\infty} \frac{g(u)}{(u-i\eta)^3}\,du \tag{A. 1.4}$$
and so on, and therefore all derivatives of I are bounded in the upper half-plane. Thus I is an analytic function in the upper half-plane.

Next, consider the behaviour of I as a tends to a point on the real axis, and again, without loss of generality, one can take this to be the point $a = 0$.

Then we write
$$I(i\eta) = I_1 + I_2,$$
where
$$I_1 = \int_{-\infty}^{-\sqrt{\eta}} + \int_{\sqrt{\eta}}^{\infty} \frac{g'(u)}{u-i\eta}\,du, \tag{A. 1.5}$$
and
$$I_2 = \int_{-\sqrt{\eta}}^{\sqrt{\eta}} \frac{g'(u)}{u-i\eta}\,du. \tag{A. 1.6}$$
Now as $\eta \to 0$,
$$I_1 \to \mathscr{P}\int_{-\infty}^{\infty} \frac{g'(u)}{u}\,du = J_0, \quad \text{say}, \tag{A. 1.7}$$
and this term is real. I_2 can be evaluated if $g(u)$ is sufficiently smooth near $u = 0$, so that
$$g'(u) = g'(0) + ug''(0) + O(u^2), \tag{A. 1.8}$$
for then
$$\int_{-\sqrt{\eta}}^{\sqrt{\eta}} \frac{g'(0)}{u-i\eta}\,du = g'(0)\int_{-1/\sqrt{\eta}}^{1/\sqrt{\eta}} \frac{dx}{x-i} \to \pi i g'(0) \tag{A. 1.9}$$
as $\eta \to 0$, while
$$\int_{-\sqrt{\eta}}^{\sqrt{\eta}} \frac{ug''(0)}{u-i\eta}\,du = g''(0)\left[\int_{-\sqrt{\eta}}^{\sqrt{\eta}} du + i\eta\int_{-\sqrt{\eta}}^{\sqrt{\eta}} \frac{du}{u-i\eta}\right] = g''(0)\,O(\sqrt{\eta}) \tag{A. 1. 10}$$
and tends to zero as $\eta \to 0$. Finally,
$$\int_{-\sqrt{\eta}}^{\sqrt{\eta}} \frac{O(u^2)}{u-i\eta}\,du \leqslant O(\sqrt{\eta})$$
and tends to zero with η. Hence
$$I(0) = J(0) + iK(0) = \mathscr{P}\int_{-\infty}^{\infty} \frac{g'(u)}{u}\,du + \pi i g'(0). \tag{A. 1. 11}$$

Appendix 2: Some series expansions for *I*-functions

To evaluate

$$I_{tt}(0) = 2\pi\int_0^\infty\int_{-1}^1\left[(1-\mu^2)\,U\frac{\partial h}{\partial U}+\frac{(1-\mu^2)^2}{\mu}\frac{\partial h}{\partial\mu}\right]U^2\,dU\,d\mu, \qquad (\text{A. }2.1)$$

an expression obtained in formula (37), we note that $h(\infty,\mu) = 0$ for all μ, and we find that

$$\int_0^\infty\int_{-1}^1(1-\mu^2)\,U^3\frac{\partial h}{\partial U}dU\,d\mu = -3\int_0^\infty\int_{-1}^1 U^2h(1-\mu^2)\,dU\,d\mu, \qquad (\text{A. }2.2)$$

after integration by parts. Therefore

$$I_{tt}(0) = 2\pi\int_0^\infty U^2\,dU\int_{-1}^1(1-\mu^2)\left[\frac{1-\mu^2}{\mu}\frac{\partial h}{\partial\mu}-3h\right]d\mu. \qquad (\text{A. }2.3)$$

On expanding $$h(U,\mu) = \sum_{n=0}^{\infty} h_{2n}(U)\,P_{2n}(\mu),$$

and with the definition $$\int_0^\infty U^2h_{2n}(U)\,dU = H_{2n},$$

we find that $$I_{tt}(0) = 2\pi\sum_{n=0}^{\infty} H_{2n}\int_{-1}^1(1-\mu^2)\left[\frac{1-\mu^2}{\mu}P'_{2n}-3P_{2n}\right]d\mu. \qquad (\text{A. }2.4)$$

The integral equals -4 when $n = 0$. We now evaluate it for other values of n. With the help of the well-known recurrence formula

$$(1-\mu)^2P'_n = n(P_{n-1}-\mu P_n),$$

we find that

$$\begin{aligned} i_n &\equiv \int_{-1}^1(1-\mu^2)\left[\frac{1-\mu^2}{\mu}P'_{2n}-3P_{2n}\right]d\mu \\ &= \int_{-1}^1 2n\left(\frac{1}{\mu}-\mu\right)P_{2n-1}-(2n+3)(1-\mu^2)\,P_{2n}\,d\mu. \end{aligned} \qquad (\text{A. }2.5)$$

Next

$$\begin{aligned}\int_{-1}^1\mu P_{2n-1}\,d\mu &= \tfrac{2}{3} \quad \text{when} \quad n = 1,\\ &= 0 \quad \text{when} \quad n > 1,\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^1 P_{2n}(1-\mu^2)\,d\mu = \frac{2}{3}\int_{-1}^1 P_{2n}(1-P_2)\,d\mu &= -\tfrac{4}{15} \quad \text{when} \quad n = 1,\\ &= 0 \quad \text{when} \quad n > 1.\end{aligned}$$

Finally to evaluate $$\int_{-1}^1\frac{P_{2n-1}(\mu)}{\mu}d\mu$$

we use the recurrence formula

$$(n+1)\,P_{n+1}-(2n+1)\,\mu P_n+nP_{n-1} = 0,$$

so that $$(2n-1)\,P_{2n-1}/\mu = (4n-3)\,P_{2n-2}-(2n-2)\,P_{2n-3}/\mu$$

and $$\int_{-1}^1\frac{P_{2n-1}}{\mu}d\mu = \frac{4n-3}{2n-1}\int_{-1}^1 P_{2n-2}\,d\mu-\frac{2n-2}{2n-1}\int_{-1}^1\frac{P_{2n-3}}{\mu}d\mu. \qquad (\text{A. }2.6)$$

For $n = 1$, $$\int_{-1}^1\frac{P_{2n-1}}{\mu}d\mu = 2, \qquad (\text{A. }2.7)$$

and therefore for larger values of n

$$\int_{-1}^{1} \frac{P_{2n-1}}{\mu} d\mu = (-1)^{n-1} \frac{(2n-2)\dots 2}{(2n-1)\dots 3} . 2$$

$$= (-1)^{n-1} \frac{[(n-1)!]^2 2^{2n-1}}{(2n-1)!}. \tag{A. 2.8}$$

On collecting terms we find that

$$i_1 = -2 . \tfrac{2}{3} + 5 . \tfrac{4}{15} + 4 = 4, \tag{A. 2.9}$$

and for $n \geqslant 2$

$$i_n = (-1)^{n-1} \frac{2^{2n} n[(n-1)!]^2}{(2n-1)!} = (-1)^{n-1} \frac{2^{2n+1}(n!)^2}{(2n)!}. \tag{A. 2.10}$$

Thus

$$I_u(0) = -8\pi H_0 + 2\pi \sum_{n=1}^{\infty} i_n H_{2n}. \tag{A. 2.11}$$

Next consider the integrals $I_v(0)$ and $I_w(0)$. In polar co-ordinates,

$$I_v(0) = -\pi i \int_0^\infty \int_0^{2\pi} \left(\frac{\partial f}{\partial \theta}\right)_{\frac{1}{2}\pi} U \cos\varphi \, dU \, d\varphi$$

$$\equiv \pi i \int_0^\infty \int_0^{2\pi} \left(\frac{\partial f}{\partial \mu}\right)_0 U \cos\varphi \, dU \, d\varphi. \tag{A. 2.12}$$

Once again let $f = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} f_{2n}^{(m)}(U) p_{2n}^m(\mu) \cos(m\varphi + \epsilon_{m,n}).$

The integration picks out the component with $m = 1$, and

$$I_v(0) = \pi^2 i \sum_{n=0}^{\infty} \left(\frac{dp_{2n}^1}{d\mu}\right)_0 \cos\epsilon_{1,n} \int_0^\infty U f_{2n}^{(1)} \, dU; \tag{A. 2.13}$$

similarly

$$I_w(0) = -\pi^2 i \sum_{n=0}^{\infty} \left(\frac{dp_{2n}^1}{d\mu}\right)_0 \sin\epsilon_{1,n} \int_0^\infty U f_{2n}^{(1)} \, dU. \tag{A. 2.14}$$

As before, we compare the two series expressions

$$f(\mathbf{u}) = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} f_{2n}^{(m)}(U) p_{2n}^m(\mu) \cos(m\varphi + \epsilon_{m,n})$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \psi_{2n}^{(m)}(U) S_{2n}^m(\alpha, \beta). \tag{A. 2.15}$$

Now consider a particular wave-normal $\mathbf{n}$, through $\alpha = \alpha_0$ and $\beta = \beta_0$, and let the azimuthal angle φ be measured from the plane which contains $\mathbf{n}$ and the fundamental initial line. Then, near $\alpha = \alpha_0$, $\beta = \beta_0$,

$$\frac{\partial}{\partial \alpha} \equiv \frac{\partial}{\partial \theta}, \quad \text{taken along } \varphi = 0,$$

$$\frac{1}{\sin\alpha} \frac{\partial}{\partial \beta} \equiv \frac{\partial}{\partial \theta}, \quad \text{taken along } \varphi = \pi/2.$$

From now on we use $\cos\theta$ as the argument of the Legendre functions. Equating harmonics of equal order in (A. 2.15),

$$\sum_{m=0}^{2n} f_{2n}^{(m)}(U)\,p_{2n}^{m}(\cos\theta)\cos m\varphi\cos\epsilon_{m,n} - \sum_{m=0}^{2n} f_{2n}^{(m)}(U)\,p_{2n}^{m}(\cos\theta)\sin m\varphi\sin\epsilon_{m,n} = \sum_{m=0}^{2n}\psi_{2n}^{(m)}(U)\,S_{2n}^{m}(\alpha,\beta). \quad \text{(A. 2.16)}$$

Hence, near $\alpha = \alpha_0$, $\beta = \beta_0$,

$$\sum_{m=0}^{2n}\psi_{2n}^{(m)}(U)\left(\frac{\partial S_{2n}^{m}}{\partial\alpha}\right)_{\alpha_0,\beta_0} = f_{2n}^{(1)}(U)\left(\frac{dp_{2n}^{1}}{d\theta}\right)_0\cos\epsilon_{1,n} \quad \text{(A. 2.17)}$$

and

$$\sum_{m=0}^{2n}\frac{\psi_{2n}^{(m)}(U)}{\sin\alpha}\left(\frac{\partial S_{2n}^{m}}{\partial\beta}\right)_{\alpha_0,\beta_0} = -f_{2n}^{(1)}(U)\left(\frac{dp_{2n}^{1}}{d\theta}\right)_0\sin\epsilon_{1,n}, \quad \text{(A. 21.8)}$$

since derivatives of the Legendre functions p_{2n}^{m} vanish at $\theta = 0$ when $m \neq 1$. Thus $I_v(0)$ and $I_w(0)$ are expressible, respectively, as infinite series, in terms of $\Xi_{2n}^{(m)}(\partial S_{2n}^{m}/\partial\alpha)_{\alpha_0,\beta_0}$ and $\Xi_{2n}^{(m)}(\partial S_{2n}^{m}/\partial\beta)_{\alpha_0,\beta_0}$ where $\Xi_{2n}^{(m)} = \int_0^\infty U\psi_{2n}^{(m)}\,dU$. In fact,

$$I_v = \pi^2 i\frac{\partial}{\partial\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{2n}\frac{(dp_{2n}^{1}/d\mu)_{\mu=0}}{(dp_{2n}^{1}/d\theta)_{\theta=0}}\Xi_{2n}^{(m)}S_{2n}^{m} \equiv \frac{\partial G}{\partial\alpha}, \quad \text{say,} \quad \text{(A. 2.19)}$$

and

$$I_w = \frac{1}{\sin\alpha}\frac{\partial G}{\partial\beta}.$$

Thus, if I_v and I_w are required to vanish for all values of α and β, then G is the sum of an infinite series, in spherical harmonics, whose first derivatives vanish everywhere on the sphere. Hence all the $\Xi_{2n}^{(m)}$ vanish and so

$$\int_0^\infty Uf(U,\alpha,\beta)\,dU$$

is independent of α and β in this case, as earlier stated.